Term Test #2
ACT 2120 – Interest Theory
Monday Oct. 23rd, 4:00 PM - 5:15 PM

There are 10 Questions worth 5 marks each.
50 marks total.
Show all your work.

$\ddot{a}_{\overline{n}|} - n v^n$

$\ddot{s}_{\overline{n}|} - n$

1.) 75 is deposited into an account at the beginning of every 6-year period for 72 years. The account credits interest at an annual effective rate of i. The accumulated value in the account at the end of 72 years is X, which is 4 times the accumulated amount at the end of 36 years. Calculate X.

$n - a_{\overline{n}|}$

$n(1+i)^n - s_{\overline{n}|}$

2.) Find the present value of a continuously increasing annuity with a term of 8 years if the force of interest is $\delta = 0.05$ and if the rate of payment at time t is t^3 per annum.

t^3

3.) For a given effective annual interest rate i, you are told that it will take 9.9696 years for a deposit to double in value. Tommy has taken out a loan in the amount of \$667.01 at this same interest rate i. Tommy will make a monthly payment of \$10 at the beginning of each month for n years at the end of which time his loan will be exactly paid off.

Compute the value of: $\frac{\partial}{\partial \delta} \ddot{a}_{\overline{n}|i}$ $\quad \frac{1 - v^n}{d}$

4.) Deposits are to be made to a fund each January 1 and July 1 for the years 1985 through 1995. The deposit made on each July 1 will be 10.25% greater than the one made on the immediately preceding January 1. The deposit made on each January 1 (except for January 1, 1985) will be the same amount as the deposit made on the immediately preceding July 1. The fund will be credited with interest at a nominal annual rate of 10%, compounded semi-annually. On December 31, 1995, the fund will have a balance of 14,000. Determine the initial deposit to the fund.

5.) At time t=0, Paul deposits P into a fund crediting interest at an effective annual interest rate of 8%. At the end of each year in years 6 through 20, Paul withdraws an amount sufficient to purchase an annuity-due of 100 per month for 10 years at a nominal interest rate of 12% compounded monthly. Immediately after the withdrawal at the end of year 20, the fund value is zero.

Calculate P.

6.) At an annual effective interest rate of i, i > 0%, the present value of a perpetuity paying 10 at the end of each 3-year period, with the first payment at the end of year 6, is 32. At the same annual effective rate of i, the present value of a perpetuity-immediate paying 1 at the end of each 4-month period is X.

Calculate X.

7.) You are the purchasing manager for your company's warehouse. All of the light-bulbs in your warehouse recently burnt out. Available for purchase are two different types of bulbs: Type A that last for 5 years and produce 6 Kilowatts of power, and Type B that last for 2 years and produce 2 Kilowatts of power. The price of Type A bulbs is decreasing by 1% each year while the price of type B bulbs decreases by 4% each year.

You are also given the following:

- You would need to keep your warehouse lit for the next 20 years.
- It takes 12000 Kilowatts of power to light your warehouse.
- The annual effective interest rate is 5.5%.
- Currently Type A bulbs cost 5 times as much as Type B.
- Type B bulbs cost $1.

Assuming that both type A and type B bulbs will be available for sale over the next 20 years how much will you save buying Type A bulbs instead of Type B in today's dollars?

8.) Bertha borrows $75,000 to be repaid over 30 years. You are given:

- Her first payment is X ~~and~~ at the end of 6 months.
- Her payments increase by $50 each 6 months for the next 8.5 years and then remain level for the following 15 years.
- After 24 years her semi-annual payment decreases by 1.5% each 6 months until the loan is paid off at time 30.
- The annual effective rate of discount is 4.5%.

Calculate X.

9.) A loan of $25,000 is to be repaid by annual payments at the end of each year for the next 20 years. During the first 5 years the payments are k per year; during the second 5 years the payments are 2k per year; during the third 5 years, 3k per year; and during the fourth 5 years; 4k per year.

If the effective annual rate of interest is $i = 12\%$, compute k.

10.) A certain Professor named Daciw has decided to move out of Winnipeg. He is tired of playing in the Winnipeg Insurer's Hockey League (WIHL) and has decided to move to Ireland to pursue a career in the Dublin Pub Fighting League (DPFL). He takes out a 25 year mortgage for $250,000 today to purchase a house in Dublin.

After being battered and bruised in Dublin for exactly 7 years he returns to Winnipeg and is looking to purchase a house and a cabin with cash.

You are also given the following information.

- The price of homes in Dublin increases by 12% each year
- The price of homes in Winnipeg decreases by .75% each quarter
- Daciw will eventually sell his new house in Winnipeg at time 20 for $125,000
- The effective interest rate is 6% for all years
- Daciw likes actuarial models
- Daciw's monthly mortgage payment in Dublin is $1586.55
- All dollars are Canadian dollars

Using only the cash left over after selling his house in Dublin and paying off the mortgage, how much can he afford to spend on his cabin in Winnipeg?

Solutions to Term Test #2

① $75\,\ddot{s}_{\overline{12}|r} = X = 4 \cdot 75 \cdot \ddot{s}_{\overline{6}|r}$ where $(1+r) = (1+i)^6$

$$75\left(\frac{(1+r)^{12}-1}{\cancel{r}}\right)\cancel{(1+r)} = 300\left(\frac{(1+r)^6-1}{\cancel{r}}\right)\cancel{(1+r)}$$

$$\Rightarrow \frac{300}{75} = \frac{(1+r)^{12}-1}{(1+r)^6-1} \qquad \text{let } x = (1+r)^6 \qquad \Rightarrow \frac{x^2-1}{x-1} = 4$$

$$\Rightarrow x^2 - 4x + 3 = 0 \qquad \therefore x = 3 \qquad (1+r)^6 = 3 \qquad r = 20.09\%$$

$$X = 75\,\ddot{s}_{\overline{12}|20.09\%} \Rightarrow X = 3586.01$$

②
$$\int_0^8 t^3 \cdot \exp\left[-\int_0^t .05\,ds\right] dt = \int_0^8 t^3 \cdot e^{-0.05t}\,dt$$

Using integration by parts:

$u = t^3$ $\quad v = \frac{-e^{-.05t}}{.05}$

$du = 3t^2$ $\quad dv = e^{-.05t}$

$$= \frac{-t^3}{.05} \cdot e^{-.05t}\Big|_0^8 + \int_0^8 3t^2 \cdot \frac{e^{-.05t}}{.05}\,dt$$

$u = t$ $\quad v = -\frac{e^{-.05t}}{.05}$

$du = 1$ $\quad dv = e^{-.05t}$

$$= -6864.08 + \frac{3}{.05}\left[\frac{-t^2}{.05} e^{-.05t}\Big|_0^8 + \int_0^8 2t \cdot \frac{e^{-.05t}}{.05}\,dt\right]$$

$$= -6864.08 - 51480.58 + \frac{3 \cdot 2}{(.05)^2}\left[\frac{-t \cdot e^{-.05t}}{.05}\Big|_0^8 + \int_0^8 \frac{e^{-.05t}}{.05}\,dt\right]$$

$$= -6864.08 - 51480.58 - 257402.90 + \frac{3.2}{.05^3}\left[-\frac{e^{-.05t}}{.05}\Big|_0^8\right]$$

$$= -315747.56 + 48000\left[6.593599\right] = \$745.20$$

(3) $1\cdot(1+i)^{9.9696} = 2$ $\qquad 667.01 = 10\cdot \ddot{a}_{\overline{n\cdot 12}|\frac{i^{(12)}}{12}}$ $\Rightarrow n = 7$ years

$\Rightarrow i = 7.2\% \Rightarrow \delta = 6.95\%$

$$\ddot{a}_{\overline{n}|i} = 1 + v + v^2 + \dots v^{n-1} = 1 + e^{-\delta} + e^{-2\delta} + \dots e^{-(n-1)\delta}$$

$$\frac{d}{d\delta}\ddot{a}_{\overline{7}|.072} = -e^{-.0695} - 2e^{-2(.0695)} - 3e^{-3(.0695)} - \dots - 6e^{-6(.0695)}$$

$$= -15.62$$

(4)

X	X(1.1025)	X(1.1025)	X(1.1025)²	X(1.1025)²	X(1.1025)³	X(1.1025)³	...	X(1.1025)¹⁰	X(1.1025)¹¹
Jan 85'	July 85'	Jan 86'	July 86'	Jan 87'	July 87'	Jan 88'	...	Jan 95'	July 95'

$i^2 = 10\%$

$\left(1 + \frac{.10}{2}\right)^2 = (1+i)$ $\quad i = 10.25\%$

$$\text{PV of all January pymts} = X + \frac{X(1.1025)}{1.1025} + \frac{X(1.1025)^2}{1.1025^2} + \dots \frac{X(1.1025)^{10}}{1.1025^{10}}$$

$$= 11\cdot X$$

$$\text{PV of all July pymts } \underline{\text{at July 84'}} = \frac{X(1.1025)}{1.1025} + \frac{X(1.1025)^2}{1.1025^2} + \dots \frac{X(1.1025)^{11}}{1.1025^{11}}$$

$$= 11\cdot X$$

PV of all July pymts at Jan 1, 1985 $= 11 \cdot X\left(1+\frac{.10}{2}\right)^1 = 11.55X$

$\therefore$ Total PV $= 11X + 11.55X = 22.55X$

$\Rightarrow 22.55X = \dfrac{14000}{\left(1+\frac{.10}{2}\right)^{22}}$ $\quad \underline{X = 212.23}$

(5) $100\, \ddot{a}_{\overline{120}|r} = 7039.75$

$r = \frac{.12}{12} = .01$

$P = 7039.753 \cdot a_{\overline{15}|8\%} \cdot v^5$

$P = 41{,}009.64$

(6) $PV = 32 = \dfrac{10}{r} \cdot \dfrac{1}{(1+r)}$

where $(1+r) = (1+i)^3$

$PV = X = \dfrac{1}{r'}$ where $(1+r') = (1+i)^{1/3}$ and $(1+r') = (1+r)^{1/9}$

$\Rightarrow 32r + 32r^2 = 10$

$32r^2 + 32r - 10 = 0 \quad r = 25\%$

$\Rightarrow (1.25)^{1/9} = 1 + r' \quad r' = 2.51\%$

$X = \dfrac{1}{.0251} = \underline{39.83}$

(7) Type A

\# of purchases required $= \frac{20}{5} = 4$ (1st purchase today, next in 5 yrs..)

\# of bulbs needed $= \frac{12000}{6} = 2000$

$price_0 = \$5$

t $(1+j) = (1+i)^5$

$j = 30.696\%$

et $(1+k') = (1+k)^5$

$(1+k') = (1-.01)^5$

$k' = -.04901$

Type B

\# of purchases required $= \frac{20}{2} = 10$

\# of bulbs needed $= \frac{12000}{2} = 6000$

$price_0 = \$1$

let $(1+r) = (1+i)^2$ $r = 11.3025\%$

let $(1+g) = (1+k)^2$

$(1+g) = (1-.04)^2$

$g = -.0784$

$$\text{Cost of Type A} = 2000 \cdot \$5 \left[\frac{1 - \frac{(1-.04901)^4}{(1+.30696)^4}}{.30696 - -.04901}\right] \cdot (1+.30696) = 2000 \cdot 5 \cdot [2.6423] = 26423.37$$

$$\text{Cost of Type B} = 6000 \cdot \$1 \left[\frac{1 - \frac{(1-.0784)^{10}}{(1+.113025)^{10}}}{.113025 - -.0784}\right] \cdot (1+.113025) = 6000 \cdot 1 \cdot [4.9336] = 29601.66$$

$\therefore$ Savings today $= 29601.66 - 26423.37 = \$\underline{3178.28}$

(8.) $(1+i) = (1-d)^{-1}$ $\quad i = 4.71204\%$

let $(1+r)^2 = (1+i)$ $\quad r = 2.328902\%$

$$75000 = \left(X \cdot a_{\overline{18|}r} + 50\left[\frac{a_{\overline{18|}r} - 18v_r^{18}}{r}\right]\right) + \left((X + 50 \cdot 17) \cdot a_{\overline{30|}r} \cdot v_r^{18}\right) +$$

$$\left((X + 50 \cdot 17) \cdot (1 + -.015) \cdot \left[\frac{1 - \frac{(1 + -.015)^{12}}{(1 + .0233)^{12}}}{.0233 - -.015}\right] \cdot v_r^{48}\right)$$

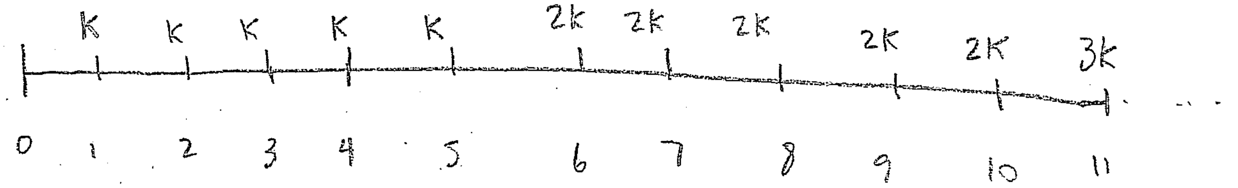
$$75000 = (14.567396 \cdot X + 5740.23) + (14.150307 \cdot X + 12027.76) +$$

$$(3.12871 \cdot X + 2659.40)$$

$$75000 = 31.846413X + 20427.39$$

$$\underline{X = 1713.62}$$

(9)



This is an increasing annuity with a payment every 5 years for 20 years. The payment amount is equal to $(K \cdot s_{\overline{5}|i})$ and Q is equal to $(K \cdot s_{\overline{5}|i})$ as well.

$$P = K \cdot s_{\overline{5}|i} = 6.352847 \cdot K = Q$$

$$PV = 25000 = (6.352847K) \cdot a_{\overline{4}|r} + (6.352847K)\left[\frac{\ddot{a}_{\overline{4}|r} - 4v_r^4}{r}\right]$$

$$= (6.352847K) \cdot (1.175763) + (6.352847 \cdot K)(.9983659)$$

$$25000 = 7.469442K + 6.3424658K$$

$$K = 1810.03$$

<u>or</u>

$$K \cdot \left(s_{\overline{20}|i} + s_{\overline{15}|i} + s_{\overline{10}|i} + s_{\overline{5}|i}\right) = 25000(1.12)^{20}$$

(10) cash from sale of house in Dublin $= 250\,000(1.12)^7 - 1586.55 \cdot a_{\overline{216}|r}$

let $(1+r)^{12} = 1+i$

$r = .48675\%$

$= \text{sale price} - PV_7 \text{ of outstanding mortgage}$

$= \$552\,670.35 - \$211\,751.69$

$= \$340\,918.66$

cost of wpg home at $t=7$ $= \dfrac{125\,000}{(1-.0075)^{4 \cdot 13}} = \$184\,894.17$

$\therefore$ Dacian can spend $340\,918.66 - 184\,894.17 = \$156\,024.49$ on his cabin.